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On the geometrical representation of the path integral reduction Jacobian: The case of dependent coordinates in the description of the reduced motion

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Abstract

The geometrical representation of the path integral reduction Jacobian obtained in the problem of the path integral quantization of a scalar particle motion on a smooth compact Riemannian manifold with the given free isometric action of the compact semisimple Lie group has been found for the case when the local reduced motion is described by means of dependent coordinates. The result is based on the scalar curvature formula for the original manifold which is viewed as a total space of the principal fibre bundle.

keywords: Marsden-Weinstein reduction, Kaluza–Klein theories, path integral, stochastic analysis.

1 Introduction

In a well-known approach to the path integral quantization of the gauge theories [1], the dynamic of true degrees of freedom, that are given by the gauge invariant variables, is presented by means of the evolution of the dependent variables defined on a gauge surface. The action of the gauge group on the space of the gauge fields, being viewed as a free isometric action of this group on the Hilbert manifold of the gauge fields, leads to the fibre bundle picture.

The choice of the local coordinates on a total space of the principal fibre bundle is performed by making use of the gauge conditions by which the local sections of the principal fibre bundle can be defined. Since in general we cannot "resolve the gauge" and determine the local coordinates on a gauge surface in terms of the explicitly definable functions, we are forced to use the dependent variables for description of the true evolution.

In path integrals, the transition to new variables and the consequent restriction of the evolution to the gauge surface, being the path integral transformations,[1] may not be a quite correct procedure. The reason of it is that at present

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[1]These transformations are called the path integral reduction.

there is no a rigorously defined path integral measure on the space of gauge fields.

In a finite-dimensional case, the quantum reduction problem in dynamical systems with a symmetry leads to the analogous path integral transformations. But in this case, the problem of the path integral reduction can be resolved [2, 3], since we know (at least for the Wiener path integrals) how to define the path integral measure and some of its transformations. Being established for the finite-dimensional dynamical systems, the rules of the path integral transformations in the reduction procedure could be extended to the path integrals used in gauge theories.

With this end in view, we have studied [3] a finite-dimensional dynamical system which is close by its properties to the gauge theories. This system describes a motion of a scalar particle on a smooth compact finite-dimensional Riemannian manifold with a given free isometric action of a compact semisimple Lie group.

The path integral reduction procedure was realized as the transformations of the original Wiener path integral, representing the diffusion (or the "quantum evolution") of a scalar particle, to the path integral which determines the "quantum evolution" of a new dynamical system (a reduced one) given on the orbit space. Our path integral transformations result in the integral relations between both path integrals. Also, a non-invariance of the path integral measure under the reduction has been found. The obtained Jacobian gives rise to the additional potential term in the Hamilton operator of the reduced dynamical system.

The purpose of the present paper is to find the geometrical representation of the reduction Jacobian. Our derivation of such a representation will be based on the formula which expresses the scalar curvature of the Riemannian manifold, which is a total space of the principal fibre bundle, in terms of the geometrical data characterizing this bundle. This formula is similar to the formula for the scalar curvature of the Riemannian manifold with the given Kaluza–Klein metric.

The paper will be organized as follows. In Section 2, we give the basic definitions and the brief review of the results obtained in [3]. Besides, in Section 3, by making use of the Itô's identity, we rewrite the exponential of the Jacobian to replace the stochastic integral of the Jacobian for the ordinary integral taken with respect to the time variable.

Next section deals with the derivation of the scalar curvature formula for our Riemannian manifold. By transforming the original coordinate basis to the horizontal lift basis,[2] we calculate the Christoffel coefficients, the Ricci curvature, and the scalar curvature.

In Section 5, by using the scalar curvature formula obtained in the previouse section, we rewrite the reduction Jacobian.

A possible application of the obtained geometrical representation of the Jacobian is discussed in Conclusion.

[2] In this nonholonomic basis the original metric takes the block-diagonal form.

2 Definitions

In [3], the diffusion of a scalar particle on a smooth compact Riemannian manifold $\mathcal{P}$ has been considered. In case of a given free isometric smooth action of a semisimple compact Lie group $\mathcal{G}$ on this manifold we can regard the manifold $\mathcal{P}$ as a total space of the principal fibre bundle $\pi : \mathcal{P} \to \mathcal{P}/\mathcal{G} = \mathcal{M}$. It means that on $\mathcal{P}$ it is possible to introduce new coordinates that are related to the fibre bundle.

The original coordinates Q^A given on a local chart of the manifold $\mathcal{P}$ have been transformed for the special coordinates (Q^{*A}, a^α) $(A = 1, \ldots, N_{\mathcal{P}}, N_{\mathcal{P}} = \dim \mathcal{P}; \alpha = 1, \ldots, N_{\mathcal{G}}, N_{\mathcal{G}} = \dim \mathcal{G})$. In order for the transformation to be a one-to-one mapping, the coordinates Q^{*A} must be subjected to the additional constraints: $\chi^\alpha(Q^*) = 0$. These constraints are chosen in such a way to define the local submanifolds in the manifold $\mathcal{P}$. We have assumed that these local submanifolds determine the global manifold Σ. Hence, our principal fibre bundle $P(\mathcal{M}, \mathcal{G})$ is trivial. Since it is locally isomorphic to the trivial bundle $\Sigma \times \mathcal{G} \to \Sigma$, the coordinates Q^{*A} can be used for the coordinatization of the manifold $\mathcal{M}$ – the base of the fibre bundle. This transformation is fulfilled at the first step of the reduction procedure.

In new coordinate basis $(\frac{\partial}{\partial Q^{*A}}, \frac{\partial}{\partial a^\alpha})$, the original metric $\tilde{G}_{\mathcal{AB}}(Q^*, a)$ of the manifold $\mathcal{P}$ becomes as follows:

$$\left(\begin{array}{cc} G_{CD}(Q^*)(P_\perp)^C_A (P_\perp)^D_B & G_{CD}(Q^*)(P_\perp)^D_A K^C_\mu \bar{u}^\mu_\alpha(a) \\ G_{CD}(Q^*)(P_\perp)^C_A K^D_\nu \bar{u}^\nu_\beta(a) & \gamma_{\mu\nu}(Q^*)\bar{u}^\mu_\alpha(a)\bar{u}^\nu_\beta(a) \end{array} \right), \tag{1}$$

where $G_{CD}(Q^*) \equiv G_{CD}(F(Q^*, e))$ is given by

$$G_{CD}(Q^*) = F^M_C(Q^*, a) F^N_D(Q^*, a) G_{MN}(F(Q^*, a)),$$

(e is an identity element of the group $\mathcal{G}$). F^A are the functions by which the right action of the group $\mathcal{G}$ on a manifold $\mathcal{P}$ is realized, $F^C_B(Q, a) \equiv \frac{\partial F^C}{\partial Q^B}(Q, a)$.

K_μ are the Killing vector fields for the Riemannian metric $G_{AB}(Q)$. In (1) they are constrained to the submanifold $\Sigma \equiv \{\chi^\alpha = 0\}$, i.e., the components K^A_μ depend on Q^*.

In (1), an orbit metric $\gamma_{\mu\nu}$ is defined by the relation $\gamma_{\mu\nu} = K^A_\mu G_{AB} K^B_\nu$.

The projection operators $P_\perp$ (depending on Q^*) is given by

$$(P_\perp)^A_B = \delta^A_B - \chi^\alpha_B (\chi\chi^\top)^{-1}{}^\beta_\alpha (\chi^\top)^A_\beta,$$

$(\chi^\top)^A_\beta$ is a transposed matrix to the matrix $\chi^\nu_B \equiv \frac{\partial \chi^\nu}{\partial Q^B}$, $(\chi^\top)^A_\mu = G^{AB}\gamma_{\mu\nu}\chi^\nu_B$. This operator projects the tangent vectors onto the gauge surface Σ.

The pseudoinverse matrix $\tilde{G}^{\mathcal{AB}}(Q^*, a)$ to the matrix (1), i.e., the matrix that satisfy

$$\tilde{G}^{\mathcal{AB}}\tilde{G}_{\mathcal{BC}} = \left(\begin{array}{cc} (P_\perp)^C_B & 0 \\ 0 & \delta^\alpha_\beta \end{array} \right),$$

is

$$\left(\begin{array}{cc} G^{EF} N^C_E N^D_F & G^{SD} N^C_S \chi^\mu_D (\Phi^{-1})^\nu_\mu \bar{v}^\sigma_\nu \\ G^{CB} \chi^\gamma_C (\Phi^{-1})^\beta_\gamma N^D_B \bar{v}^\alpha_\beta & G^{CB} \chi^\gamma_C (\Phi^{-1})^\beta_\gamma \chi^\mu_B (\Phi^{-1})^\nu_\mu \bar{v}^\alpha_\beta \bar{v}^\sigma_\nu \end{array} \right). \tag{2}$$

Here $(\Phi^{-1})^{\beta}_{\mu}$ – the matrix which is inverse to the Faddeev – Popov matrix:

$$(\Phi)^{\beta}_{\mu}(Q)=K^A_{\mu}(Q)\frac{\partial\chi^{\beta}(Q)}{\partial Q^A}.$$

In (2), the asymmetric projection operator N,

$$N^A_C=\delta^A_C-K^A_{\alpha}(\Phi^{-1})^{\alpha}_{\mu}\chi^{\mu}_C,$$

onto the orthogonal to the Killing vector field subspace, has the following properties:

$$N^A_B N^B_C=N^A_C,\quad (P_{\perp})^{\tilde{A}}_B N^C_{\tilde{A}}=(P_{\perp})^C_B,\qquad N^{\tilde{A}}_B(P_{\perp})^C_{\tilde{A}}=N^C_B.$$

The matrix $\bar{v}^{\alpha}_{\beta}(a)$ is inverse to the matrix $\bar{u}^{\alpha}_{\beta}(a)$. The $\det\bar{u}^{\alpha}_{\beta}(a)$ is a density of a right invariant measure given on the group $\mathcal{G}$.

The determinant of the matrix (1) is equal to

$$\begin{aligned}(\det\tilde{G}_{\mathcal{AB}})&=\det G_{AB}(Q^*)\det\gamma_{\alpha\beta}(Q^*)(\det\chi\chi^{\top})^{-1}(Q^*)(\det\bar{u}^{\mu}_{\nu}(a))^2\\&\quad\times(\det\Phi^{\alpha}_{\beta}(Q^*))^2\det(P_{\perp})^C_B(Q^*)\\&=\det\Big((P_{\perp})^D_A\;G^{\rm H}_{DC}\;(P_{\perp})^C_B\Big)\det\gamma_{\alpha\beta}\;(\det\bar{u}^{\mu}_{\nu})^2.\end{aligned}$$

It does not vanish only on the surface Σ. On this surface $\det(P_{\perp})^C_B$ is equal to unity.

Note also that the "horizontal metric" $G^{\rm H}$ is defined by the relation $G^{\rm H}_{DC}=\Pi^{\tilde{D}}_D\,\Pi^{\tilde{C}}_C\,G_{\tilde{D}\tilde{C}}$ in which $\Pi^A_B=\delta^A_B-K^A_{\mu}\gamma^{\mu\nu}K^D_{\nu}G_{DB}$ is the projection operator.

An original diffusion of a scalar particle on a smooth compact Riemannian manifold $\mathcal{P}$ has been described by the backward Kolmogorov equation

$$\begin{cases}\left(\dfrac{\partial}{\partial t_a}+\dfrac{1}{2}\mu^2\kappa\triangle_{\mathcal{P}}(p_a)+\dfrac{1}{\mu^2\kappa m}V(p_a)\right)\psi_{t_b}(p_a,t_a)=0\\ \psi_{t_b}(p_b,t_b)=\phi_0(p_b), \qquad\qquad (t_b>t_a).\end{cases}\tag{3}$$

In this equation $\mu^2=\frac{\hbar}{m}$, κ is a real positive parameter, $\triangle_{\mathcal{P}}(p_a)$ is a Laplace–Beltrami operator on manifold $\mathcal{P}$, and $V(p)$ is a group–invariant potential term. In a chart with the coordinate functions $Q^A=\varphi^A(p)$, the Laplace – Beltrami operator has the standard form:

$$\triangle_{\mathcal{P}}(Q)=G^{-1/2}(Q)\frac{\partial}{\partial Q^A}G^{AB}(Q)G^{1/2}(Q)\frac{\partial}{\partial Q^B},$$

where $G=det(G_{AB})$, $G_{AB}(Q)=G(\frac{\partial}{\partial Q^A},\frac{\partial}{\partial Q^B})$.

We have used the definition of the path integrals from [4] for representing the solution of the equation (3). This solution is written as follows:

$$\begin{aligned}\psi_{t_b}(p_a,t_a)&=\mathrm{E}\Big[\phi_0(\eta(t_b))\exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}V(\eta(u))du\}\Big]\\&=\int_{\Omega_-}d\mu^{\eta}(\omega)\phi_0(\eta(t_b))\exp\{\ldots\},\end{aligned}\tag{4}$$

where $\eta(t)$ is a global stochastic process on a manifold $\mathcal{P}$. $\Omega_-=\{\omega(t):\omega(t_a)=0,\eta(t)=p_a+\omega(t)\}$ is the path space on this manifold. The path integral in

measure μ^η is defined by the probability distribution of a stochastic process $\eta(t)$.

Since the global semigroup determined by the equation (4) is defined by the limit of the superposition of the local semigroups

$$\psi_{t_b}(p_a,t_a)=U(t_b,t_a)\phi_0(p_a)=\lim_q\tilde{U}_\eta(t_a,t_1)\cdot\ldots\cdot\tilde{U}_\eta(t_{n-1},t_b)\phi_0(p_a), \qquad (5)$$

we derive the transformation properties of the path integral of (4) by studying the local semigroups $\tilde{U}_\eta$. These local semigroup are given by the path integrals with the integration measures determined by the local representatives $\eta^A(t)$ of the global stochastic process $\eta(t)$. The local processes $\eta^A(t)$ are solutions of the stochastic differential equations:

$$d\eta^A(t)=\frac{1}{2}\mu^2\kappa G^{-1/2}\frac{\partial}{\partial Q^B}(G^{1/2}G^{AB})dt+\mu\sqrt{\kappa}\mathfrak{X}^A_{\bar{M}}(\eta(t))dw^{\bar{M}}(t), \qquad (6)$$

where the matrix $\mathfrak{X}^A_{\bar{M}}$ is defined by the local equality $\sum_{\bar{K}=1}^{n_P}\mathfrak{X}^A_{\bar{K}}\mathfrak{X}^B_{\bar{K}}=G^{AB}$. (We denote the Euclidean indices by over–barred indices.)

A replacement of the coordinates Q^A for (Q^{*A},a^α) does not change the path integral measures in the local semigroups as this transformation is related to the phase space transformation of the stochastic processes.

The second step of the reduction procedure consists of the factorization of the path integral measure. The local evolution given on the orbit has been separated from the evolution on the orbit space and we have come to the integral relation between the original path integral and the corresponding path integral for the evolution on the orbit space. The last evolution has been written in terms of the dependent coordinates.

In particular case of the reduction performed onto the zero–momentum level, the $\lambda=0$ case of the general formula from [3], the integral relation between the path integrals for the Green's functions is

$$G_\Sigma(Q^*_b,t_b;Q^*_a,t_a)=\int_{\mathcal{G}}G_{\mathcal{P}}(p_b\theta,t_b;p_a,t_a)d\mu(\theta), \quad (Q^*=\pi_\Sigma(p)). \qquad (7)$$

The path integral for the Green's function $G_{\mathcal{P}}$ may be obtained from the path integral (4) by choosing the delta-function as an initial function. The Green's function G_Σ is presented by the following path integral

$$\begin{aligned}G_\Sigma(Q^*_b,t_b;Q^*_a,t_a)&=\int_{\substack{\xi_\Sigma(t_a)=Q^*_a\\ \xi_\Sigma(t_b)=Q^*_b}}d\mu^{\xi_\Sigma}\exp\Big\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}V(\xi_\Sigma(u))du\Big\}\\ &\times\exp\int_{t_a}^{t_b}\Big\{-\frac{1}{2}\mu^2\kappa\left[(P_\perp)^D_A\,G^{\rm H}_{DL}\,(P_\perp)^L_B\right]j^A_{II}\,j^B_{II}dt\\ &+\mu\sqrt{\kappa}\,G^{\rm H}_{DL}(P_\perp)^D_A\,j^A_{II}\,\tilde{\mathfrak{X}}^L_{\bar{M}}\,dw^{\bar{M}}_t\Big\},\end{aligned} \qquad (8)$$

where $j^A_{II}(Q^*)$ is the projection of the mean curvature vector of the orbit on the submanifold Σ. This vector has two equal representations[3]

$$\begin{aligned}j^A_{II}(Q^*)&=-\frac{1}{2}\,G^{EU}N^A_E N^D_U\left[\gamma^{\alpha\beta}G_{CD}(\tilde{\nabla}_{K_\alpha}K_\beta)^C\right](Q^*)\\ &=-\frac{1}{2}\,N^A_C\left[\gamma^{\alpha\beta}(\tilde{\nabla}_{K_\alpha}K_\beta)^C\right](Q^*),\end{aligned} \qquad (9)$$

[3] In [3] this vector was written with a wrong sign.

where

$$(\tilde{\nabla}_{K_\alpha}K_\beta)^C(Q^*)=K^A_\alpha(Q^*)\,\frac{\partial}{\partial Q^A}K^C_\beta(Q)\Big|_{Q=Q^*}+K^A_\alpha(Q^*)K^B_\beta(Q^*)\tilde{\Gamma}^C_{AB}(Q^*)$$

with

$$\tilde{\Gamma}^C_{AB}(Q^*)=\frac{1}{2}\,G^{CE}(Q^*)\Big(\frac{\partial}{\partial Q^{*A}}G_{EB}(Q^*)+\frac{\partial}{\partial Q^{*B}}G_{EA}(Q^*)-\frac{\partial}{\partial Q^{*E}}G_{AB}(Q^*)\Big).$$

In the path integral (8), the path integral measure is determined by the stochastic process ξ_Σ. Its local stochastic differerential equations are as follows

$$dQ^{*A}_t=\mu^2\kappa\bigg(-\frac{1}{2}G^{EM}N^C_EN^B_M\,{}^H\Gamma^A_{CB}+j^A_I\bigg)dt+\mu\sqrt{\kappa}N^A_C\tilde{\mathfrak{X}}^C_{\bar{M}}dw^{\bar{M}}_t, \tag{10}$$

where j_I is the mean curvature vector of the orbit space. The Christoffel symbols ${}^H\Gamma^B_{CD}$ in (10) are defined by the equality

$$G^H_{AB}\ {}^H\Gamma^B_{CD}=\frac{1}{2}\left(G^H_{AC,D}+G^H_{AD,C}-G^H_{CD,A}\right), \tag{11}$$

in which by the derivatives we mean the following: $G^H_{AC,D}\equiv\frac{\partial G^H_{AC}(Q)}{\partial Q^D}\Big|_{Q=Q^*}$.

We note that the special form of the stochastic differential equation (10) results from the fact that the orbit space can be viewed as a submanifold of the (Riemannian) manifold $(\mathcal{P},G^H_{AB}(Q))$ with the degenerate metric G^H_{AB}.

The semigroup determined by the path integral (8) acts in the space of the scalar functions given on Σ. The differential generator (the Hamilton operator) of this semigroup is

$$\frac{1}{2}\mu^2\kappa\left\{G^{CD}N^A_CN^B_D\frac{\partial^2}{\partial Q^{*A}\partial Q^{*B}}-G^{CD}N^E_CN^M_D\,{}^H\Gamma^A_{EM}\frac{\partial}{\partial Q^{*A}}\right.$$
$$\left.+\left(j^A_I+j^A_{II}\right)\frac{\partial}{\partial Q^{*A}}\right\}+\frac{1}{\mu^2\kappa m}\tilde{V}. \tag{12}$$

3 Transformation of the stochastic integral

In the integrand of the path integral (8) there is a term with the Itô's stochastic integral. It is not difficult to get rid of this integral by making use of the Itô's identity. But first we should rewrite the second exponent function[4] standing at the integrand of the path integral (8). From the properties of introduced projection operators it follows that this function can be rewritten as

$$\exp\Big\{-\frac{1}{8}\mu^2\kappa\int_{t_a}^{t_b}G^{\mathrm{H}}_{CB}\gamma^{\nu\sigma}(\nabla_{K_\nu}K_\sigma)^C\gamma^{\alpha\beta}(\nabla_{K_\alpha}K_\beta)^B dt$$
$$-\frac{1}{2}\mu\sqrt{\kappa}\int_{t_a}^{t_b}G^{\mathrm{H}}_{CD}\gamma^{\nu\sigma}(\nabla_{K_\nu}K_\sigma)^C\tilde{\mathfrak{X}}^D_{\bar{M}}dw^{\bar{M}}_t\Big\}. \tag{13}$$

[4]It is the Jacobian of the performed Girsanov transformation.

By using the Itô's differentiation formula from the stochastic calculus, it can be shown the following equality:

$$
\begin{aligned}
&{\rm e}^{\sigma(Q^*(t))} = {\rm e}^{\sigma(Q^*(t_a))}\\
&\qquad\times\,{\rm e}^{\mu\sqrt{\kappa}\int\limits_{t_a}^{t}\frac{\partial\sigma}{\partial Q^{*C}}B^C_{\bar M}dw^{\bar M}_t+\mu^2\kappa\int\limits_{t_a}^{t}\left(\frac{1}{2}\frac{\partial^2\sigma}{\partial Q^{*A}{Q^{*C}}}B^A_{\bar M}B^C_{\bar M}+\frac{\partial\sigma}{\partial Q^{*C}}a^C\right)dt}, \qquad (14)
\end{aligned}
$$

provided that the stochastic variable Q^{*A}_t satisfies the stochastic differential equation

$$
dQ^{*A}_t = a^A(Q^*_t)\,dt + B^A_{\bar M}(Q^*_t)\,dw^{\bar M}_t.
$$

Eq.(14) leads to the Itô's identity by which

$$
\begin{aligned}
&{\rm e}^{\mu\sqrt{\kappa}\int\limits_{t_a}^{t_b}\left(\frac{\partial\sigma}{\partial Q^{*C}}\right)B^C_{\bar M}dw^{\bar M}_t} =\\
&\qquad\left(\frac{{\rm e}^{\sigma(Q^*(t_b))}}{{\rm e}^{\sigma(Q^*(t_a))}}\right){\rm e}^{-\mu^2\kappa\int\limits_{t_a}^{t_b}\left(\frac{1}{2}\frac{\partial^2\sigma}{\partial Q^{*A}\,{Q^{*C}}}B^A_{\bar M}B^C_{\bar M}+\frac{\partial\sigma}{\partial Q^{*C}}a^C\right)dt}. \qquad (15)
\end{aligned}
$$

In order that this equality may be applied to our case, i.e., when the stochastic variable Q^*_t satisfies the equation (10), the integrand of the stochastic integral in (13) must be appropriately transformed. It may be done by making use of the identity

$$
\gamma^{\sigma\mu}(\nabla_{K_\mu}K_\sigma)^E(Q^*) = -\frac{1}{2}G^{PE}N^A_P\Bigl(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*A}}\,\gamma_{\alpha\beta}\Bigr)(Q^*)
$$

and taking into account the following properties of the projection operators: $G^{\rm H}_{KC}G^{PC} = \Pi^P_K$ and $\Pi^P_K N^A_P = N^A_K$. That is, in (15) we may put

$$
\sigma = \frac{1}{2}\ln\det\gamma_{\alpha\beta},
$$

and $B^A_{\bar M} = N^A_L\tilde{\mathfrak{X}}^L_{\bar M}$. Also, we note that $\frac{\partial\sigma}{\partial Q^{*C}} = \frac{1}{2}\bigl(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*C}}\,\gamma_{\alpha\beta}\bigr)$.

In our case, on the right-hand side of (15) we have the integral with the following integrand:

$$
\begin{aligned}
&\frac{1}{4}G^{\tilde A\tilde B}N^A_{\tilde A}N^B_{\tilde B}\frac{\partial}{\partial Q^{*A}}\Bigl(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*B}}\,\gamma_{\alpha\beta}\Bigr)\\
&\qquad+\frac{1}{2}\Bigl(-\frac{1}{2}G^{EM}N^P_EN^S_M\,{}^{\rm H}\Gamma^A_{PS}+j^A_{\rm I}\Bigr)\Bigl(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*A}}\,\gamma_{\alpha\beta}\Bigr). \qquad (16)
\end{aligned}
$$

Since the mean curvature vector $j^A_{\rm I}$ of the orbit space is equal to

$$
j^A_{\rm I} = \frac{1}{2}G^{\tilde B\tilde D}N^B_{\tilde B}N^D_{\tilde D}\Bigl(N^A_{BD}+{}^{\rm H}\Gamma^A_{BD}-N^A_C\,{}^{\rm H}\Gamma^C_{BD}\Bigr),
$$

($N^A_{BD}\equiv\frac{\partial}{\partial Q^{*D}}N^A_B$), (16) can be transformed into

$$
\begin{aligned}
&\frac{1}{4}G^{\tilde A\tilde B}N^A_{\tilde A}N^B_{\tilde B}\frac{\partial}{\partial Q^{*A}}\Bigl(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*B}}\,\gamma_{\alpha\beta}\Bigr)\\
&\qquad+\frac{1}{4}G^{\tilde B\tilde D}N^B_{\tilde B}N^D_{\tilde D}\Bigl(N^A_{BD}-N^A_C\,{}^{\rm H}\Gamma^C_{BD}\Bigr)\Bigl(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*A}}\,\gamma_{\alpha\beta}\Bigr). \qquad (17)
\end{aligned}
$$

In the obtained expression, the second component of the sum may be further simplified. $G^{\tilde{B}\tilde{D}}N^B_{\tilde{B}}N^D_{\tilde{D}}N^A_{BD}$ can be rewritten as

$$G^{\tilde{B}\tilde{D}}N^B_{\tilde{B}}N^D_{\tilde{D}}N^A_{BD}=G^{\tilde{B}\tilde{D}}N^D_{\tilde{D}}N^A_{\tilde{B}D}-G^{\tilde{B}\tilde{D}}K^B_{\mu}\Lambda^{\mu}_{\tilde{B}}N^D_{\tilde{D}}N^A_{BD} \tag{18}$$

and

$$G^{\tilde{B}\tilde{D}}N^B_{\tilde{B}}N^D_{\tilde{D}}N^A_C\,{}^{\mathrm{H}}\Gamma^C_{BD}=G^{\tilde{B}D}N^B_{\tilde{B}}N^A_C\,{}^{\mathrm{H}}\Gamma^C_{BD}-G^{\tilde{B}\tilde{D}}N^B_{\tilde{B}}K^D_{\mu}\Lambda^{\mu}_{\tilde{D}}N^A_C\,{}^{\mathrm{H}}\Gamma^C_{BD}. \tag{19}$$

By using the identities $K^B_{\mu}N^A_{BD}=-K^B_{\mu D}N^A_B$ and $N^A_C(K^C_{\alpha D}+K^B_{\alpha}\,{}^{\mathrm{H}}\Gamma^C_{DB})=0$, one can show that the last terms of the equalities (18) and (19) are equal. Hence, they don't make a contribution to (17). We obtain, therefore, the following expression for the integrand:

$$\begin{aligned}&\frac{1}{4}G^{\tilde{A}\tilde{B}}N^A_{\tilde{A}}N^B_{\tilde{B}}\frac{\partial}{\partial Q^{*A}}\Big(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*B}}\gamma_{\alpha\beta}\Big)\\&\quad+\frac{1}{4}\Big(G^{\tilde{B}\tilde{D}}N^D_{\tilde{D}}N^A_{\tilde{B}D}-G^{\tilde{B}D}N^B_{\tilde{B}}N^A_C\,{}^{\mathrm{H}}\Gamma^C_{BD}\Big)\Big(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*A}}\gamma_{\alpha\beta}\Big).\end{aligned}$$

Thus, after application of the Itô's identity in (13), the path integral reduction Jacobian can be rewriten as follows:

$$\Big(\frac{\gamma(Q^*(t_b))}{\gamma(Q^*(t_a))}\Big)^{\frac{1}{4}}\exp\Big\{-\frac{1}{8}\mu^2\kappa\int_{t_a}^{t_b}\tilde{J}dt\Big\}, \tag{20}$$

where

$$\begin{aligned}\tilde{J}=&\frac{1}{4}G^{PB}N^A_BN^E_P\Big(\gamma^{\mu\nu}\frac{\partial}{\partial Q^{*A}}\gamma_{\mu\nu}\Big)\Big(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*E}}\gamma_{\alpha\beta}\Big)\\&+G^{\tilde{A}\tilde{B}}N^A_{\tilde{A}}N^B_{\tilde{B}}\frac{\partial}{\partial Q^{*A}}\Big(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*B}}\gamma_{\alpha\beta}\Big)\\&+\Big(G^{\tilde{C}\tilde{A}}N^F_{\tilde{A}}N^B_{\tilde{C}F}-G^{\tilde{A}\tilde{C}}N^E_{\tilde{A}}N^B_M\,{}^{\mathrm{H}}\Gamma^M_{E\tilde{C}}\Big)\Big(\gamma^{\alpha\beta}\frac{\partial}{\partial Q^{*B}}\gamma_{\alpha\beta}\Big).\end{aligned} \tag{21}$$

The main task of the present paper is to get the geometrical representation for the integrand $\tilde{J}$. It will be done with the formula for the Riemannian curvature scalar of $\mathcal{P}$.

4 The scalar curvature of the bundle

The scalar curvature of the Riemannian manifold $\mathcal{P}$ will be calculated by using the special nonholonomic basis. This basis generalizes the horizontal lift basis considered in [5]. It consists of the horizontal vector fields H_A and the left-invariant vector fields $L_{\alpha}=v^{\mu}_{\alpha}(a)\frac{\partial}{\partial a^{\mu}}$. The vector fields L_{α} have the standard commutation relations

$$[L_{\alpha},L_{\beta}]=c^{\gamma}_{\alpha\beta}L_{\gamma},$$

where the $c^{\gamma}_{\alpha\beta}$ are the structure constant of the group $\mathcal{G}$.

The vector fields H_A are given by

$$H_A=N^E_A(Q^*)\left(\frac{\partial}{\partial Q^{*E}}-\tilde{\mathcal{A}}^{\alpha}_E\,L_{\alpha}\right),$$

where $\tilde{\mathcal{A}}^{\alpha}_{E}(Q^{*},a)=\bar{\rho}^{\alpha}_{\mu}(a)\,\mathcal{A}^{\mu}_{E}(Q^{*})$. The matrix $\bar{\rho}^{\alpha}_{\mu}$ is inverse to the matrix ρ^{β}_{α} of the adjoint representation of the group $\mathcal{G}$, and $\mathcal{A}^{\nu}_{P}=\gamma^{\nu\mu}K^{R}_{\mu}\,G_{RP}$ is the "mechanical" connection defined on our principal fibre bundle.

The commutation relation of the horizontal vector fields are

$$[H_C,H_D]=(\Lambda^{\gamma}_{C}N^{P}_{D}-\Lambda^{\gamma}_{D}N^{P}_{C})K^{S}_{\gamma P}\,H_S-N^{E}_{C}N^{P}_{D}\,\tilde{\mathcal{F}}^{\alpha}_{EP}L_{\alpha},$$

where $\Lambda^{\gamma}_{D}=(\Phi^{-1})^{\gamma}_{\mu}\,\chi^{\mu}_{D}$. The curvature $\tilde{\mathcal{F}}^{\alpha}_{EP}$ of the connection $\tilde{\mathcal{A}}$ is given by

$$\tilde{\mathcal{F}}^{\alpha}_{EP}=\frac{\partial}{\partial Q^{*E}}\,\tilde{\mathcal{A}}^{\alpha}_{P}-\frac{\partial}{\partial Q^{*P}}\,\tilde{\mathcal{A}}^{\alpha}_{E}+c^{\alpha}_{\nu\sigma}\,\tilde{\mathcal{A}}^{\nu}_{E}\,\tilde{\mathcal{A}}^{\sigma}_{P},$$

$(\tilde{\mathcal{F}}^{\alpha}_{EP}(Q^{*},a)=\bar{\rho}^{\alpha}_{\mu}(a)\,\mathcal{F}^{\mu}_{EP}(Q^{*})\,)$. In calculation of the commutation relation we have used the following equality:

$$L_{\alpha}\,\tilde{\mathcal{A}}^{\lambda}_{E}=-c^{\lambda}_{\alpha\mu}\,\tilde{\mathcal{A}}^{\mu}_{E},$$

which comes from the equation satisfied by ρ: $L_{\alpha}\,\rho^{\gamma}_{\beta}=c^{\mu}_{\alpha\beta}\,\rho^{\gamma}_{\mu}$.

The above commutation relation may be also written as follows:

$$[H_C,H_D]=\mathcal{C}^{A}_{CD}\,H_A+\mathcal{C}^{\alpha}_{CD}L_{\alpha},$$

where the structure constants of the nonholonomic basis are

$$\mathcal{C}^{A}_{CD}=(\Lambda^{\gamma}_{C}K^{A}_{\gamma D}-\Lambda^{\gamma}_{D}K^{A}_{\gamma C})$$

and

$$\mathcal{C}^{\alpha}_{CD}=-N^{S}_{C}N^{P}_{D}\,\tilde{\mathcal{F}}^{\alpha}_{SP}\,.$$

In our basis, L_{α} commutes with H_A:

$$[H_A,L_{\alpha}]=0.$$

In the horizontal lift basis, the metric (1) can be written as

$$\check{G}_{\mathcal{A}\mathcal{B}}=\begin{pmatrix} G^{\mathrm{H}}_{AB} & 0\\ 0 & \tilde{\gamma}_{\alpha\beta}\end{pmatrix},\tag{22}$$

with

$$\tilde{G}(H_A,H_B)\equiv G^{\mathrm{H}}_{AB}(Q^{*}),\quad \tilde{G}(L_{\alpha},L_{\beta})\equiv\tilde{\gamma}_{\alpha\beta}(Q^{*},a)=\gamma_{\alpha'\beta'}(Q^{*})\,\rho^{\alpha'}_{\alpha}(a)\,\rho^{\beta'}_{\beta}(a).$$

The dual basis is given by the following one-forms:

$$\omega^{A}=(\mathrm{P}_{\perp})^{A}_{S}\,dQ^{*S},$$
$$\omega^{\alpha}=u^{\alpha}_{\mu}da^{\mu}+\tilde{\mathcal{A}}^{\alpha}_{E}\,(\mathrm{P}_{\perp})^{E}_{S}\,dQ^{*S},$$

for which $\omega^{A}(H_B)=N^{A}_{B}$, $\omega^{\alpha}(H_A)=0$, and $\omega^{\alpha}(L_{\beta})=\delta^{\alpha}_{\beta}$.

The pseudoiverse matrix $\check{G}^{\mathcal{A}\mathcal{B}}$ to the matrix (22) is defined by

$$\check{G}^{\mathcal{A}\mathcal{B}}=\begin{pmatrix} G^{EF}N^{A}_{E}N^{B}_{F} & 0\\ 0 & \tilde{\gamma}^{\alpha\beta}\end{pmatrix},$$

with

$$\tilde{G}(\omega^{A},\omega^{B})\equiv G^{EF}N^{A}_{E}N^{B}_{F},\ \ \tilde{G}(\omega^{\alpha},\omega^{\beta})\equiv\tilde{\gamma}^{\alpha\beta}=\gamma^{\alpha'\beta'}\,\bar{\rho}^{\alpha}_{\alpha'}\bar{\rho}^{\beta}_{\beta'},\ \ \tilde{G}(\omega^{A},\omega^{\alpha})=0.$$

The orthogonality condition is

$$\check{G}^{\mathcal{A}\mathcal{B}}\check{G}_{\mathcal{B}\mathcal{C}}=\begin{pmatrix} N^{A}_{C} & 0\\ 0 & \delta^{\alpha}_{\beta}\end{pmatrix}.$$

4.1 The Christoffel symbols

The computation of the connection coefficients $\check{\Gamma}^{\mathcal{D}}_{\mathcal{A}\mathcal{B}}$ in the nonholonomic basis will be preformed by using the following formula:

$$
\begin{aligned}
2\,\check{\Gamma}^{\mathcal{D}}_{\mathcal{A}\mathcal{B}}\,\tilde{G}(\partial_{\mathcal{D}},\partial_{\mathcal{C}}) = \partial_{\mathcal{A}}\,\tilde{G}(\partial_{\mathcal{B}},\partial_{\mathcal{C}}) + \partial_{\mathcal{B}}\,\tilde{G}(\partial_{\mathcal{A}},\partial_{\mathcal{C}}) - \partial_{\mathcal{C}}\,\tilde{G}(\partial_{\mathcal{A}},\partial_{\mathcal{B}}) \\
-\tilde{G}(\partial_{\mathcal{A}},[\partial_{\mathcal{B}},\partial_{\mathcal{C}}]) - \tilde{G}(\partial_{\mathcal{B}},[\partial_{\mathcal{A}},\partial_{\mathcal{C}}]) + \tilde{G}(\partial_{\mathcal{C}},[\partial_{\mathcal{A}},\partial_{\mathcal{B}}]),
\end{aligned}
\tag{23}
$$

where the terms of the form $\partial_{\mathcal{A}}\tilde{G}$ denote the corresponding directional derivatives.

First we consider the calculation of the coordinate components $\check{\Gamma}^{A}_{BC}$. In this case (23) can be rewritten as

$$
2\check{\Gamma}^{D}_{AB}\,G^{\rm H}_{DC} = H_A G^{\rm H}_{BC} + H_B G^{\rm H}_{AC} - H_C G^{\rm H}_{AB} - G^{\rm H}_{AP}\mathcal{C}^{P}_{BC} - G^{\rm H}_{BP}\mathcal{C}^{P}_{AC} + G^{\rm H}_{CP}\mathcal{C}^{P}_{AB}.
$$

Replacing structure constant $\mathcal{C}^{A}_{BC}$ by their explicit expressions and performing the necessary transformations, we get

$$
\check{\Gamma}^{D}_{AB}\,G^{\rm H}_{DC} = N^{E}_{A}\,{}^{\rm H}\Gamma_{BEC} - N^{E}_{B}\,{}^{\rm H}\Gamma_{ACE} + N^{E}_{C}\,{}^{\rm H}\Gamma_{ABE} + {}^{\rm H}\Gamma_{ACB} - {}^{\rm H}\Gamma_{BAC}, \tag{24}
$$

where

$$
{}^{\rm H}\Gamma_{ABC} = G^{\rm H}_{AC,B} + G^{\rm H}_{BC,A} - G^{\rm H}_{AB,C}.
$$

Eq.(24) was obtained by making use of the equality $G^{\rm H}_{AP}K^{P}_{\mu C} = -K^{P}_{\mu}G^{\rm H}_{AP,C}$, which can be derived by means of the differentiation (with respect to Q^*) of the relation $G^{\rm H}_{AP}K^{P}_{\mu} = 0$.

From the Killing identity for the metric G_{AB} it follows that

$$
K^{E}_{\mu}G^{\rm H}_{AB,E} = 0.
$$

Taking this equality into account we transform eq.(24) into

$$
G^{\rm H}_{DC}\,\check{\Gamma}^{D}_{AB} = N^{E}_{A}\,{}^{\rm H}\Gamma_{BEC}.
$$

Multiplying the both sides of this equation on $G^{SF}N^{P}_{S}N^{C}_{F}$ and using the identity $G^{MS}N^{A}_{M}\,{}^{\rm H}\Gamma_{CDS} = N^{A}_{S}\,{}^{\rm H}\Gamma^{S}_{CD}$, we get

$$
N^{P}_{S}\,\check{\Gamma}^{S}_{AB} = N^{P}_{K}N^{E}_{A}\,{}^{\rm H}\Gamma^{K}_{BE}.
$$

Hence, we can write

$$
\check{\Gamma}^{D}_{AB} = N^{E}_{A}\,{}^{\rm H}\Gamma^{D}_{BE}
$$

(modulo the terms X^{S}_{ABC} for which $N^{P}_{S}X^{S}_{ABC} = 0$).

The calculation of other non-vanishing coordinate components of the Christoffel connection $\check{\Gamma}^{\mathcal{D}}_{\mathcal{A}\mathcal{B}}$ on $\mathcal{P}$ leads to the following result:

$$
\begin{aligned}
\check{\Gamma}^{\mu}_{AB} &= -\frac{1}{2}N^{E}_{A}N^{F}_{B}\,\tilde{\mathcal{F}}^{\mu}_{EF}, \\
\check{\Gamma}^{P}_{\alpha B} &= \frac{1}{2}G^{PS}N^{F}_{S}N^{E}_{B}\,\tilde{\mathcal{F}}^{\mu}_{EF}\tilde{\gamma}_{\mu\alpha}, \\
\check{\Gamma}^{P}_{A\beta} &= \frac{1}{2}G^{PS}N^{F}_{S}N^{E}_{A}\,\tilde{\mathcal{F}}^{\mu}_{EF}\tilde{\gamma}_{\mu\beta}, \\
\check{\Gamma}^{P}_{\alpha\beta} &= -\frac{1}{2}\,G^{PS}H_S\tilde{\gamma}_{\alpha\beta} = -\frac{1}{2}G^{PS}N^{E}_{S}\,\tilde{\mathcal{D}}_{E}\tilde{\gamma}_{\alpha\beta},
\end{aligned}
$$

$$\check{\Gamma}^{\mu}_{\alpha B} = \frac{1}{2}\tilde{\gamma}^{\mu\nu} H_B \tilde{\gamma}_{\alpha\nu} = \frac{1}{2}\tilde{\gamma}^{\mu\nu} N^E_B\, \tilde{\mathfrak{D}}_E \tilde{\gamma}_{\alpha\nu},$$
$$\check{\Gamma}^{\mu}_{A\beta} = \frac{1}{2}\tilde{\gamma}_{\mu\nu} H_A \tilde{\gamma}_{\beta\nu} = \frac{1}{2}\tilde{\gamma}^{\mu\nu} N^E_A\, \tilde{\mathfrak{D}}_E \tilde{\gamma}_{\beta\nu},$$
$$\check{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2}\tilde{\gamma}^{\mu\nu}(c^{\sigma}_{\alpha\beta}\tilde{\gamma}_{\sigma\nu} - c^{\sigma}_{\nu\beta}\tilde{\gamma}_{\alpha\sigma} - c^{\sigma}_{\nu\alpha}\tilde{\gamma}_{\beta\sigma}).$$

In these formulae the covariant derivatives are given as follows:

$$\tilde{\mathfrak{D}}_E \tilde{\gamma}_{\alpha\beta} = \Big(\frac{\partial}{\partial Q^{*E}}\tilde{\gamma}_{\alpha\beta} - c^{\sigma}_{\mu\alpha}\tilde{\mathcal{A}}^{\mu}_E \tilde{\gamma}_{\sigma\beta} - c^{\sigma}_{\mu\beta}\tilde{\mathcal{A}}^{\mu}_E \tilde{\gamma}_{\sigma\alpha}\Big).$$

4.2 The Ricci curvature tensor

To evaluate the components $\check{R}_{\mathcal{AC}}$ of the the Ricci curvature tensor of the metric (22) relative to our nonholonomic basis we make use of the following formula:

$$\check{R}_{\mathcal{AC}} = \partial_{\mathcal{A}}\check{\Gamma}^{\mathcal{P}}_{\mathcal{PC}} - \partial_{\mathcal{P}}\check{\Gamma}^{\mathcal{P}}_{\mathcal{AC}} + \check{\Gamma}^{\mathcal{D}}_{\mathcal{PC}}\check{\Gamma}^{\mathcal{P}}_{\mathcal{AD}} - \check{\Gamma}^{\mathcal{E}}_{\mathcal{AC}}\check{\Gamma}^{\mathcal{P}}_{\mathcal{PE}} - \mathcal{C}^{\mathcal{E}}_{\mathcal{AP}}\check{\Gamma}^{\mathcal{P}}_{\mathcal{EC}}. \tag{25}$$

For the components $\check{R}_{AC}$ this formula can be written as

$$\begin{aligned}\check{R}_{AC} \;&=\; H_A\check{\Gamma}^M_{MC} - H_M\check{\Gamma}^M_{AC} + \check{\Gamma}^K_{MC}\check{\Gamma}^M_{AK} - \check{\Gamma}^E_{AC}\check{\Gamma}^K_{KE} - \mathcal{C}^K_{AM}\check{\Gamma}^M_{KC}\\ &+\; H_A\check{\Gamma}^{\mu}_{\mu C} - L_{\alpha}\check{\Gamma}^{\alpha}_{AC} + \check{\Gamma}^{\mu}_{MC}\check{\Gamma}^M_{A\mu} + \check{\Gamma}^K_{\mu C}\check{\Gamma}^{\mu}_{AK} + \check{\Gamma}^{\nu}_{\mu C}\check{\Gamma}^{\mu}_{A\nu} - \check{\Gamma}^E_{AC}\check{\Gamma}^{\nu}_{\nu E}\\ &-\; \check{\Gamma}^{\mu}_{AC}\check{\Gamma}^K_{K\mu} - \check{\Gamma}^{\mu}_{AC}\check{\Gamma}^{\nu}_{\nu\mu} - \mathcal{C}^{\alpha}_{AM}\check{\Gamma}^M_{\alpha C}.\end{aligned} \tag{26}$$

First of all, let us consider the expression standing at the first line of the right–hand side of (26). By using obtained Christoffel symbols $\check{\Gamma}$ and the coefficients $\mathcal{C}^K_{AM}$, we present this expression as follows:

$$N^S_A N^E_M \left(\frac{\partial}{\partial Q^{*S}}{}^{\mathrm{H}}\Gamma^M_{CE} - \frac{\partial}{\partial Q^{*E}}{}^{\mathrm{H}}\Gamma^M_{CS} + {}^{\mathrm{H}}\Gamma^K_{CE}\,{}^{\mathrm{H}}\Gamma^M_{KS} - {}^{\mathrm{H}}\Gamma^P_{CS}\,{}^{\mathrm{H}}\Gamma^M_{PE}\right).$$

It may also be rewritten as

$$N^S_A N^E_M\,{}^{\mathrm{H}}R_{SEC}{}^M,$$

where by ${}^{\mathrm{H}}R_{SEC}{}^M$ we denote the expression which looks like the Riemann curvature tensor of the manifold with the degenerate metric G^{H}_{AB}.

We note that our Christoffel symbols ${}^{\mathrm{H}}\Gamma^M_{CS}$ are defined up to the terms T^M_{CS} that satisfy the equality $G^{\mathrm{H}}_{AB}T^B_{CD} = 0$. Therefore, we are allowed to neglect the terms that are directly proportional to K^B_{α}. Since in our case

$$N^E_M\,{}^{\mathrm{H}}R_{SEC}{}^M = {}^{\mathrm{H}}R_{SMC}{}^M - K^E_{\alpha}\Lambda^{\alpha}_M\,{}^{\mathrm{H}}R_{SEC}{}^M,$$

the contribution to the Ricci curvature $\check{R}_{AC}$ coming from the second term of the previous expression may be omitted. The first term will give us $N^S_A\,{}^{\mathrm{H}}R_{SMC}{}^M$.

The remaining terms of $\check{R}_{AC}$ are presented by the following expressions:

$$\begin{aligned}H_A\,{}^{\mathrm{H}}\check{\Gamma}^{\nu}_{\nu C} \;&=\; \frac{1}{2}H_A(\tilde{\gamma}^{\mu\nu}H_C\tilde{\gamma}_{\mu\nu}) \equiv \frac{1}{2}N^F_A\frac{\partial}{\partial Q^{*F}}\left(\gamma^{\mu\nu}N^E_C\frac{\partial}{\partial Q^{*E}}\gamma_{\mu\nu}\right);\\ L_{\alpha}\check{\Gamma}^{\alpha}_{AC} \;&=\; -\frac{1}{2}N^E_A N^P_C\, L_{\alpha}\tilde{\mathcal{F}}^{\alpha}_{EP};\end{aligned}$$

$$\begin{aligned}
\check{\Gamma}^{\nu}_{MC}\check{\Gamma}^{M}_{A\nu} &= -\frac{1}{4}(G^{MS}N^{E}_{M}N^{P}_{S})\,N^{F}_{C}N^{Q}_{A}\,\tilde{\mathcal{F}}^{\mu}_{EF}\tilde{\mathcal{F}}^{\nu}_{QP}\tilde{\gamma}_{\mu\nu};\\
\check{\Gamma}^{K}_{\nu C}\check{\Gamma}^{\nu}_{AK} &= -\frac{1}{4}(G^{KS}N^{F}_{S}N^{R}_{K})\,N^{E}_{A}N^{P}_{C}\,\tilde{\mathcal{F}}^{\nu}_{PF}\tilde{\mathcal{F}}^{\mu}_{ER}\tilde{\gamma}_{\mu\nu};\\
\check{\Gamma}^{\mu}_{\alpha C}\check{\Gamma}^{\alpha}_{A\mu} &= \frac{1}{4}(\tilde{\gamma}^{\mu\nu}H_{C}\tilde{\gamma}_{\alpha\nu})\,(\tilde{\gamma}^{\alpha\beta}H_{A}\tilde{\gamma}_{\mu\beta})\\
&= \frac{1}{4}\tilde{\gamma}^{\mu\nu}N^{E}_{C}\big(\tilde{\mathfrak{D}}_{E}\tilde{\gamma}_{\alpha\nu}\big)\,\tilde{\gamma}^{\alpha\beta}N^{F}_{A}\big(\tilde{\mathfrak{D}}_{F}\tilde{\gamma}_{\mu\beta}\big);\\
\check{\Gamma}^{E}_{AC}\check{\Gamma}^{\nu}_{\nu E} &= \frac{1}{2}N^{P}_{A}\,{}^{\mathrm{H}}\Gamma^{E}_{CP}\,\big(\tilde{\gamma}^{\mu\nu}H_{E}\tilde{\gamma}_{\mu\nu}\big).
\end{aligned}$$

In derivation of these terms we have used the identity $c^{\sigma}_{\sigma\mu}=0$, which is valid for the compact semisimple Lie group.

Collecting the parts of $\check{R}_{AC}$, we get

$$\begin{aligned}
\check{R}_{AC} &= N^{S}_{A}N^{E}_{M}\,{}^{\mathrm{H}}R_{SEC}{}^{M} - \frac{1}{2}N^{E}_{A}N^{F}_{C}\,L_{\alpha}\tilde{\mathcal{F}}^{\alpha}_{EF} + \frac{1}{2}N^{E}_{A}N^{F}_{C}\,\tilde{\mathcal{F}}^{\alpha}_{EF}\check{\Gamma}^{\nu}_{\nu\alpha}\\
&+ \frac{1}{2}(G^{MS}N^{P}_{M}N^{F}_{S})\,N^{E}_{A}N^{R}_{C}\,\tilde{\mathcal{F}}^{\alpha}_{EP}\tilde{\mathcal{F}}^{\mu}_{RF}\tilde{\gamma}_{\mu\alpha} - \frac{1}{2}N^{P}_{A}\,{}^{\mathrm{H}}\Gamma^{E}_{CP}(\tilde{\gamma}^{\mu\nu}H_{E}\,\tilde{\gamma}_{\mu\nu})\\
&+ \frac{1}{2}H_{A}\,\big(\tilde{\gamma}^{\mu\nu}H_{C}\,\tilde{\gamma}_{\mu\nu}\big) + \frac{1}{4}\big(\tilde{\gamma}^{\mu\nu}H_{C}\,\tilde{\gamma}_{\alpha\nu}\big)\big(\tilde{\gamma}^{\alpha\beta}H_{A}\,\tilde{\gamma}_{\mu\beta}\big). \qquad (27)
\end{aligned}$$

The components $\check{R}_{\alpha\beta}$ of the Ricci curvature $\check{R}_{\mathcal{AC}}$ are defined as

$$\check{R}_{\alpha\beta} = \hat{\partial}_{\alpha}\check{\Gamma}^{\mathcal{K}}_{\mathcal{K}\beta} - \hat{\partial}_{\mathcal{K}}\check{\Gamma}^{\mathcal{K}}_{\alpha\beta} + \check{\Gamma}^{\mathcal{E}}_{\mathcal{K}\beta}\check{\Gamma}^{\mathcal{K}}_{\alpha\mathcal{E}} - \check{\Gamma}^{\mathcal{E}}_{\alpha\beta}\check{\Gamma}^{\mathcal{K}}_{\mathcal{K}\mathcal{E}} - \mathfrak{C}^{\mathcal{E}}_{\alpha\mathcal{K}}\check{\Gamma}^{\mathcal{K}}_{\mathcal{E}\beta}.$$

In our case $\check{R}_{\alpha\beta}$ are given by the following formula:

$$\begin{aligned}
\check{R}_{\alpha\beta} &= \tilde{R}_{\alpha\beta} + \frac{1}{4}\big(G^{ES}N^{F}_{S}N^{B}_{E}\big)\,\big(G^{MQ}N^{P}_{M}N^{A}_{Q}\big)\,\tilde{\gamma}_{\mu\beta}\tilde{\gamma}_{\nu\alpha}\,\tilde{\mathcal{F}}^{\mu}_{PF}\tilde{\mathcal{F}}^{\nu}_{BA}\\
&+\frac{1}{2}\,H_{M}\big(G^{MS}H_{S}\tilde{\gamma}_{\alpha\beta}\big) - \frac{1}{4}\big(\tilde{\gamma}^{\sigma\nu}H_{M}\tilde{\gamma}_{\sigma\beta}\big)\,\big(G^{MS}H_{S}\tilde{\gamma}_{\alpha\nu}\big)\\
&-\frac{1}{4}\,\big(G^{ES}H_{S}\tilde{\gamma}_{\nu\beta}\big)\,\big(\tilde{\gamma}^{\sigma\nu}H_{E}\tilde{\gamma}_{\sigma\alpha}\big) + \frac{1}{2}\,N^{Q}_{M}\,{}^{\mathrm{H}}\Gamma^{M}_{EQ}\big(G^{ES}H_{S}\tilde{\gamma}_{\alpha\beta}\big)\\
&+\frac{1}{4}\big(G^{ES}H_{S}\tilde{\gamma}_{\alpha\beta}\big)\big(\tilde{\gamma}^{\mu\nu}H_{E}\tilde{\gamma}_{\mu\nu}\big), \qquad (28)
\end{aligned}$$

in which by $\tilde{R}_{\alpha\beta}$ we denote the Ricci curvature of the manifold with the Riemannian metric $\tilde{\gamma}_{\mu\nu}$.

4.3 The calculation of the scalar curvature

In the horizontal lift basis the scalar curvature of the original manifold $\mathcal{P}$ is defined by

$$R_{\mathcal{P}} = G^{\tilde{A}\tilde{C}}N^{A}_{\tilde{A}}N^{C}_{\tilde{C}}\,\check{R}_{AC} + \tilde{\gamma}^{\alpha\beta}\check{R}_{\alpha\beta}. \qquad (29)$$

Notice that by the symmetry argument the second and the third terms of (27) will not make the contributions into $R_{\mathcal{P}}$.

First we consider the contribution to $R_{\mathcal{P}}$ which is obtained from the terms belonging to $\check{R}_{AC}$ and $\check{R}_{\alpha\beta}$ that are given with a single multiplier $(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_{A}\tilde{\gamma}_{\mu\nu})$. In (27), they are the fifth and sixth terms.

It can be shown that the sixth term of $\check{R}_{AC}$ leads to the following contribution:

$$\frac{1}{2}G^{\tilde{A}\tilde{C}}N^E_{\tilde{A}}N^B_{\tilde{C}}\,\tilde{\mathfrak{D}}_E\big(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_B\tilde{\gamma}_{\mu\nu}\big)+\frac{1}{2}G^{\tilde{A}\tilde{C}}N^E_{\tilde{A}}N^C_{\tilde{C}}N^A_{CE}\big(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_A\tilde{\gamma}_{\mu\nu}\big). \tag{30}$$

Combining (30) with the contribution of the fifth term to $R_{\mathcal{P}}$, we get

$$\frac{1}{2}G^{\tilde{A}\tilde{C}}N^{P'}_{\tilde{A}}N^{C'}_{\tilde{C}}\left(N^A_{C'P'}-N^A_E\,{}^{\mathrm{H}}\Gamma^A_{C'P'}\right)(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_A\tilde{\gamma}_{\mu\nu}). \tag{31}$$

Now we will calculate the corresponding contribution to $R_{\mathcal{P}}$ originated from the terms of $\check{R}_{\alpha\beta}$. But before proceeding to calculation these terms must be transformed. First we rewrite the third term of (28) as

$$H_M\big(G^{MS}\big)\big(H_S\tilde{\gamma}_{\alpha\beta}\big)+G^{MS}N^E_MN^P_{SE}\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\alpha\beta}+G^{MS}N^E_MN^P_S\tilde{\mathfrak{D}}_E\big(\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\alpha\beta}\big). \tag{32}$$

Then, differentiating the identity

$$N^A_L=G^{\mathrm{H}}_{LF}\big(G^{MS}N^F_MN^A_S\big)$$

with respect to Q^{*E}, we get the following equality:

$$\begin{aligned}&\big(G^{MS}\big)_{,\,E}N^E_MN^A_S=\\&\quad-\big(G^{MS}N^F_MN^A_S\big)N^E_B\,{}^{\mathrm{H}}\Gamma^B_{FE}-G^{LU}N^A_BN^E_U\,{}^{\mathrm{H}}\Gamma^B_{LE}-G^{MS}N^F_{ME}N^E_FN^A_S.\end{aligned}$$

By making use of this equality, it can be shown that the contribution of (32) into $R_{\mathcal{P}}$ are given by the following expression:

$$\begin{aligned}&\frac{1}{2}\big(-G^{MS}N^F_MN^A_SN^E_B\,{}^{\mathrm{H}}\Gamma^B_{FE}-G^{LU}N^A_BN^E_U\,{}^{\mathrm{H}}\Gamma^B_{LE}-G^{MS}N^F_{ME}N^E_FN^A_S\\&+G^{MS}N^E_MN^A_{SE}+G^{FS}N^A_SN^E_B\,{}^{\mathrm{H}}\Gamma^B_{EF}\big)(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_A\tilde{\gamma}_{\mu\nu}).\end{aligned} \tag{33}$$

Using $N^F_M=\delta^F_M-K^F_\mu\Lambda^\mu_M$ for the projector N^F_M in the first term of (33), we see that the part of the first term and the last term of this expression are mutually cancelled. Besides, grouping the third term in (33) with the remnant of the first term, we also come to zero. It take place because of the identity

$$N^C_P(K^P_{\alpha E}+K^F_\alpha\,{}^{\mathrm{H}}\Gamma^P_{FE})=0$$

which is derived from the Killing relation for the horizontal metric G^{H}_{AB}:

$$K^E_{\mu B}G^{\mathrm{H}}_{AE}+K^E_{\mu A}G^{\mathrm{H}}_{BE}+K^E_\mu G^{\mathrm{H}}_{AB,E}=0.$$

Notice that before using the identity in (33), one should make a replacement of $N^E_FN^F_{ME}$ for $-N^E_PK^P_{\nu E}\Lambda^\nu_M$ in the third term.

The second and fourth terms of (33) give us

$$\frac{1}{2}\,G^{LU}N^E_U\big(N^A_{LE}-N^A_B\,{}^{\mathrm{H}}\Gamma^B_{LE}\big)(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_A\tilde{\gamma}_{\mu\nu}). \tag{34}$$

The obtained expression is the contribution to $R_{\mathcal{P}}$ given by the terms of $\check{R}_{\alpha\beta}$.

Taking a sum of (31) and (34), we get the contribution to $R_{\mathcal{P}}$ which is obained from $\check{R}_{AC}$ and $\check{R}_{\alpha\beta}$:

$$
\begin{aligned}
&G^{\tilde{A}C'} N^{P'}_{\tilde{A}} \big(N^{A}_{C'P'} - N^{A}_{E}{}^{\mathrm{H}}\Gamma^{B}_{C'P'}\big)\big(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_A\tilde{\gamma}_{\mu\nu}\big)\\
&\qquad \equiv G^{\tilde{A}C'} N^{P'}_{\tilde{A}} \big({}^{\mathrm{H}}\nabla_{P'} N^{A}_{C'}\big)\big(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_A\tilde{\gamma}_{\mu\nu}\big).
\end{aligned}
\tag{35}
$$

Finally, we consider the contribution to $R_{\mathcal{P}}$ which is obtained from the terms belonging to $\check{R}_{AC}$ and $\check{R}_{\alpha\beta}$, and containing the product of two multipliers of the aforementioned kind.

The terms of $\check{R}_{AC}$ give the following expression as the contribution to $R_{\mathcal{P}}$:

$$
\frac{1}{2} G^{\tilde{A}\tilde{C}} N^{E}_{\tilde{A}} N^{B}_{\tilde{C}}\, \tilde{\mathfrak{D}}_E\big(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_B\tilde{\gamma}_{\mu\nu}\big) + \frac{1}{4} G^{\tilde{A}\tilde{C}} N^{A}_{\tilde{A}} N^{C}_{\tilde{C}}\, \big(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_C\tilde{\gamma}_{\alpha\nu}\big)\big(\tilde{\gamma}^{\alpha\beta}\tilde{\mathfrak{D}}_A\tilde{\gamma}_{\mu\beta}\big).
$$

The contribution from the terms of $\check{R}_{\alpha\beta}$ can be presented as follows:

$$
\begin{aligned}
&\frac{1}{2} G^{MS} N^{E}_{M} N^{P}_{S} \tilde{\gamma}^{\alpha\beta}\tilde{\mathfrak{D}}_E\big(\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\alpha\beta}\big) - \frac{1}{4} G^{MS} N^{E}_{M} N^{P}_{S} \big(\tilde{\gamma}^{\nu\sigma}\tilde{\mathfrak{D}}_E\tilde{\gamma}_{\sigma\beta}\big)\big(\tilde{\gamma}^{\alpha\beta}\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\alpha\nu}\big)\\
&-\frac{1}{4} G^{ES} N^{P}_{S} N^{R}_{E}\Big[\big(\tilde{\gamma}^{\alpha\beta}\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\nu\beta}\big)\big(\tilde{\gamma}^{\nu\sigma}\tilde{\mathfrak{D}}_R\tilde{\gamma}_{\alpha\sigma}\big) - \frac{1}{4}\big(\tilde{\gamma}^{\alpha\beta}\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\alpha\beta}\big)\big(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_R\tilde{\gamma}_{\mu\nu}\big)\Big].
\end{aligned}
$$

Replacing the first term of the last expression with the help of the equality

$$
\tilde{\gamma}^{\alpha\beta}\tilde{\mathfrak{D}}_E\big(\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\alpha\beta}\big) = \tilde{\gamma}^{\alpha\sigma}\tilde{\gamma}^{\beta\kappa}\big(\tilde{\mathfrak{D}}_E\tilde{\gamma}_{\sigma\kappa}\big)\big(\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\alpha\beta}\big) + \tilde{\mathfrak{D}}_E\big(\tilde{\gamma}^{\alpha\beta}\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\alpha\beta}\big),
$$

we add together the above contributions (from $\check{R}_{AC}$ and $\check{R}_{\alpha\beta}$) and get

$$
\begin{aligned}
&G^{\tilde{A}\tilde{C}} N^{E}_{\tilde{A}} N^{B}_{\tilde{C}} \tilde{\mathfrak{D}}_E\big(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_B\tilde{\gamma}_{\mu\nu}\big) + \frac{1}{4} G^{ES} N^{P}_{S} N^{R}_{E}\big(\tilde{\gamma}^{\alpha\beta}\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\nu\beta}\big)\big(\tilde{\gamma}^{\nu\sigma}\tilde{\mathfrak{D}}_R\tilde{\gamma}_{\alpha\sigma}\big)\\
&+\frac{1}{4} G^{ES} N^{P}_{S} N^{R}_{E}\big(\tilde{\gamma}^{\alpha\beta}\tilde{\mathfrak{D}}_P\tilde{\gamma}_{\alpha\beta}\big)\big(\tilde{\gamma}^{\mu\nu}\tilde{\mathfrak{D}}_R\tilde{\gamma}_{\mu\nu}\big).
\end{aligned}
\tag{36}
$$

Using (27), (28), together with (35) and (36), in (29), we obtain the following representation for the scalar curvature:

$$
\begin{aligned}
R_{\mathcal{P}} &= G^{A'C'} N^{S}_{A'} N^{C}_{C'} N^{E}_{M}\, {}^{\mathrm{H}}R_{SEC}{}^{M} + \tilde{\gamma}^{\alpha\beta}\tilde{R}_{\alpha\beta}\\
&+\frac{1}{4}\big(G^{ES} N^{F}_{S} N^{B}_{E}\big)\big(G^{MQ} N^{P}_{M} N^{A}_{Q}\big)\,\tilde{\gamma}_{\mu\nu}\,\tilde{\mathcal{F}}^{\mu}_{PF}\tilde{\mathcal{F}}^{\nu}_{AB} + G^{A'C'} N^{E}_{A'}\big({}^{\mathrm{H}}\tilde{\nabla}_E(\tilde{\gamma}^{\mu\nu} H_{C'}\tilde{\gamma}_{\mu\nu})\big)\\
&+\frac{1}{4} G^{ES}\big(\tilde{\gamma}^{\alpha\beta} H_S\tilde{\gamma}_{\nu\beta}\big)\big(\tilde{\gamma}^{\nu\sigma} H_E\tilde{\gamma}_{\alpha\sigma}\big) + \frac{1}{4} G^{ES}\big(\tilde{\gamma}^{\alpha\beta} H_S\tilde{\gamma}_{\alpha\beta}\big)\big(\tilde{\gamma}^{\mu\nu} H_E\tilde{\gamma}_{\mu\nu}\big).
\end{aligned}
\tag{37}
$$

Here we have used the definition

$$
{}^{\mathrm{H}}\tilde{\nabla}_E f_C \equiv \tilde{\mathfrak{D}}_E\, f_C - {}^{\mathrm{H}}\Gamma^{M}_{CE} f_M.
$$

Notice that $R_{\mathcal{P}}$ is independent of the point in the fiber where it is evaluated. It follows from the invariance of the original Riemannian metric on $\mathcal{P}$ under the action of the group $\mathcal{G}$. So, in (37) one can omit the tilde-marks placed over the letters.

5 The geometrical representation of $\tilde{J}$

By comparing the expression for $\tilde{J}$ given by (21) and (37), it can be found that $R_{\mathcal{P}}$ has the following representation:

$$R_{\mathcal{P}} = {}^{\mathrm{H}}R + R_{\mathcal{G}} + \frac{1}{4}\mathcal{F}^2 + \tilde{J} + \frac{1}{4}G^{ES}N^A_S N^B_E \gamma^{\alpha'\beta'}\gamma^{\nu'\mu'}(\mathcal{D}_A\gamma_{\nu'\beta'})(\mathcal{D}_B\gamma_{\mu'\alpha'}), \quad (38)$$

with the evident symbolical notations for

$$\begin{aligned} &{}^{\mathrm{H}}R \equiv G^{A'C'}N^S_{A'}N^C_{C'}N^E_M\,{}^{\mathrm{H}}R_{SEC}{}^M, \\ &\mathcal{F}^2 \equiv \left(G^{ES}N^F_S N^B_E\right)\left(G^{MQ}N^P_M N^A_Q\right)\gamma_{\mu\nu}\,\mathcal{F}^{\mu}_{PF}\mathcal{F}^{\nu}_{AB}, \end{aligned}$$

and for the scalar curvature of the orbit

$$R_{\mathcal{G}} \equiv \frac{1}{2}\gamma^{\mu\nu}c^{\sigma}_{\mu\alpha}c^{\alpha}_{\nu\sigma} + \frac{1}{4}\gamma_{\mu\sigma}\gamma^{\alpha\beta}\gamma^{\epsilon\nu}c^{\mu}_{\epsilon\alpha}c^{\sigma}_{\nu\beta}.$$

The last term of (38), as it will be shown, is related to the second fundamental form of the orbit.

It follows from the fact that every orbit of the group action can be locally viewed as a submanifold in the manifold $\mathcal{P}$. In this case the second fundamental form of the orbit may be defined as follows:

$$j^C_{\alpha\beta}(Q) = \Pi^C_D(Q)\left(\nabla_{K_\alpha}K_\beta\right)^D(Q),$$

where by ∇_A we denote the covariant derivative determined by means of the Levy–Civita connection of the manifold $\mathcal{P}$ with the Riemannian metric $G_{AB}(Q)$.

We must project the second fundamental form $j^C_{\alpha\beta}(Q)$ onto the direction which is parallel to the orbit space. In order to find this projection we should calculate the following expression:

$$\tilde{G}^{AB}\tilde{G}\left(\Pi^C_D(Q)\left(\nabla_{K_\alpha}K_\beta\right)^D\frac{\partial}{\partial Q^C}, \frac{\partial}{\partial Q^{*A}}\right), \quad (39)$$

where $\tilde{G}$ is the metric (1) of the manifold $\mathcal{P}$, and where before performing the calculation, the variables Q^A in

$$\Pi^C_A(Q)\left(\nabla_{K_\alpha}K_\beta\right)^A(Q)\frac{\partial}{\partial Q^C} = \frac{1}{2}\Pi^C_A(Q)\left[\nabla_{K_\alpha}K_\beta + \nabla_{K_\beta}K_\alpha\right]^A(Q)\frac{\partial}{\partial Q^C}$$

must be replaced for (Q^{*A}, a^{α}).

As a result of the calculation we get

$$j^B_{\alpha\beta}(Q^*, a) = \frac{1}{2}\rho^{\alpha'}_{\alpha}(a)\rho^{\beta'}_{\beta}(a)N^B_E(Q^*)\left(\nabla_{K_{\alpha'}}K_{\beta'} + \nabla_{K_{\beta'}}K_{\alpha'}\right)^E(Q^*).$$

Moreover, it can be shown that

$$j^B_{\alpha\beta}(Q^*, a) = -\frac{1}{2}\rho^{\mu}_{\alpha}(a)\rho^{\nu}_{\beta}(a)\,G^{PS}(Q^*)N^B_P(Q^*)N^E_S(Q^*)\left(\mathcal{D}_E\gamma_{\mu\nu}\right)(Q^*).$$

After restriction of the obtained expression to the surface Σ by setting $a = e$, where e is the unity element of the group $\mathcal{G}$, we come to the following expression for the second fundamental form:

$$j^B_{\alpha\beta}(Q^*) = -\frac{1}{2}G^{PS}N^B_P N^E_S\left(\mathcal{D}_E\gamma_{\alpha\beta}\right)(Q^*).$$

Using this expression, one can show that the last term of (38) is the "square" of the fundamental form of the orbit:

$$||j||^2 = G^{\mathrm{H}}_{AB}\,\gamma^{\alpha\mu}\,\gamma^{\beta\nu}\,j^A_{\alpha\beta}\,j^B_{\mu\nu}.$$

Thus, the integrand $\tilde{J}$ is given by

$$\tilde{J} = R_{\mathcal{P}} - {}^{\mathrm{H}}R - R_{\mathcal{G}} - \frac{1}{4}\mathcal{F}^2 - ||j||^2. \tag{40}$$

Now, we may rewrite the integral relation (7) in the following form:

$$\gamma(Q^*_b)^{-1/4}\gamma(Q^*_a)^{-1/4}G_{\mathcal{M}}(Q^*_b,t_b;Q^*_a,t_a) = \int_{\mathcal{G}} G_{\mathcal{P}}(p_b\theta,t_b;p_a,t_a)d\mu(\theta),$$

where

$$G_{\mathcal{M}}(Q^*_b,t_b;Q^*_a,t_a) = \int d\mu^{\xi_\Sigma}\exp\Bigl\{\int_{t_a}^{t_b}\Bigl[\frac{1}{\mu^2\kappa m}\tilde{V}(\xi_\Sigma(u)) - \frac{1}{8}\mu^2\kappa m\,\tilde{J}(\xi_\Sigma(u))\Bigr]du\Bigr\}.$$

The semigroup determined by the Green's function $G_{\mathcal{M}}$ acts in the Hilbert space with the scalar product $(\psi_1,\psi_2) = \int_\Sigma \psi_1(Q^*)\psi_2(Q^*)\,dv_{\mathcal{M}}(Q^*)$. The measure $dv_{\mathcal{M}}$ is given by $dv_{\mathcal{M}}(Q^*) = \det^{1/2}\Bigl((P_\perp)^D_A\,G^{\mathrm{H}}_{DC}\,(P_\perp)^C_B\Bigr)dQ^{*1}\wedge\ldots\wedge dQ^{*N_{\mathcal{P}}}$.

If it were possible to find invariant coordinates x^i such that $\chi^\alpha(Q^*(x^i))\equiv 0$, the measure $dv_{\mathcal{M}}$ of the previouse scalar product could be transformed into the volume measure $\det^{1/2}h_{ij}\;dx^1\cdot\ldots\cdot dx^{N_{\mathcal{M}}}$ for the Riemannian metric $h_{ij} = Q^{*A}_{\ i}(x)G^{\mathrm{H}}_{AB}(Q^*(x))Q^{*B}_{\ j}(x)$ defined on the orbit space $\mathcal{M}$.

The Green's function $G_{\mathcal{M}}$ satisfies the forward Kolmogorov equation with the operator

$$\hat{H}_\kappa = \frac{\hbar\kappa}{2m}\Bigl\{G^{CD}N^A_C N^B_D\frac{\partial^2}{\partial Q^{*A}\partial Q^{*B}} - G^{CD}N^E_C N^M_D\,{}^H\Gamma^A_{EM}\frac{\partial}{\partial Q^{*A}}\Bigr\} - \frac{\hbar\kappa}{8m}\tilde{J} + \frac{1}{\hbar\kappa}\tilde{V}, \tag{41}$$

where $\tilde{J}$ is given by (40). The Hamilton operator $\hat{H}$ of the corresponding Schrödinger equation can be obtained from (41) as follows: $\hat{H} = -\frac{\hbar}{\kappa}\hat{H}_\kappa|_{\kappa=i}$.

6 Conclusion

In the paper, it has been shown that the exponential of the path integral reduction Jacobian can be written in the form of the difference between the scalar curvature of the original manifold and the following terms: the scalar curvature of the orbit, the scalar curvature of the reduced manifold, the square of the second fundamental form of the orbit, and the one fourth of the square of the curvature of the connection defined on the principal fibre bundle.

In many important cases the local description of the reduced motion is only possible by making use of dependent coordinates. This is a typical situation which one meets with in gauge theories. It would be very useful to find an appropriate generalization of the obtained formula (40) in these cases.

Besides, the formulae of this kind are necessary for consideration of the renormalization corrections in case of the rigorous definition of the path integral measure defined on the space of gauge connections, where the regularization of the original (weak) metric converts it into the (strong) Riemannian metric [6].

In the paper the geometrical representation of the Jacobian has been found, in fact, for the case of the local reduction since consideration of [3] has been done for the trivial principal fibre bundle. An interesting problem would be to extend the obtained result onto the case of the global path integral reduction in which the topological questions would be expected to play an important role.

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